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A GENERALIZATION OF SNELL'S LAW

by

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A Generalization of Snell's Law

by

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ABSTRACT

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I. INTRODUCTION

Snell's Law of refraction and the principle of reflection have both been known for several hundred years. The intent of this paper is to develop some generalizations of these observed phenomena. We derive these generalizations by imposing certain constraints upon the problem and show that in all cases our general results reproduce the now familiar forms of the Law of Refraction and of the reflection principle when suitably restricted.

In this paper we make use of variational methods along with techniques from optimal control theory to develop a generalization of Snell's Law of Refraction in which the velocity of light in the two media depends upon direction alone. We also develop several interesting inverse problems which involve the determination of various indices of refraction corresponding to certain families of curves. For instance, if we desire to move from a point A to a point B along a cycloid, in order to avoid an obstruction in our path, we can determine the necessary index of refraction needed to do so.

"The Dutch astronomer and mathematician Willebrord Snell (1591-1626) discovered the law of refraction by considering a ray of light that passes from one medium, in which light travels at a velocity v_1, to a second medium, in which light travels at a velocity v_2." [Ref. 1].

Fermat's Principle states that the time elapsed in the passage of light between any two fixed points of a medium is an extremum with respect to possible paths connecting the points [Ref. 2]. Since the distance travelled is proportional to the time within each medium, the path of minimum time in a medium which has a constant velocity of propagation is a straight line. If however, we are given two media in contact with propagation velocities v_1 and v_2 ($v_1 \neq v_2$), the path followed between a point (X_1,Y_1) in medium 1 and a point (X_2,Y_2) in medium 2 consists of two line segments meeting at a point (X_0,Y_0) on the interface of the two media. This "bending" of the light ray is called refraction, and hence, Snell's Law of Refraction, which is given by:

$$\frac{\sin(\phi_1)}{v_1} = \frac{\sin(\phi_2)}{v_2}$$

where Φ_1 and Φ_2 are the angles formed by the ray and the *normal to the interface*. The paths in which light particles travel are represented by rays, which indicate the direction of propagation. These rays are perpendicular to the wave fronts, regardless of isotropy. An isotropic medium is one in which the properties of the medium are the same in all directions from any given point. A medium is homogeneous if its properties do not vary from point to point. Wavefronts can have many shapes. If the disturbances are propagated in a single direction, the waves are called plane waves. "At a given instant, conditions are the same everywhere on any plane perpendicular to the direction of propagation." [Ref. 1:p. 301].

It has been observed that if a plane light wave falls on another plane surface the light beam is reflected from the surface [Ref. 1]. The angle of incidence, Φ_1, and the angle of reflection, Φ_3, measured from the normal to the surface satisfy the principle of reflection which is given by:

$$\Phi_1 = \Phi_3$$

In the study of acoustic wave propagation, Snell's Law of Refraction and the law governing the reflection principle are usually derived from the reduced Helmholtz equation. Although this thesis concerns generalizations of these laws using variational methods, a brief outline of this process is included to illustrate other available techniques.

Figure 1.1 depicts plane wave reflection and refraction from a planar interface. The equation for the incident plane wave, P_i is given by:

$$P_i = e^{ik_u \vec{x}\cdot\vec{d}_i} = e^{ik_u(x\sin\theta_i - y\cos\theta_i)}$$

where the incident amplitude has been normalized to 1, the vector $\vec{x}$ is a position vector, and $\vec{d}$ is the vector which points in the direction of propagation of the wave. Using subscripts u for the upper medium, l for the lower medium, i for incident, r for reflected, and t for transmitted we have the following system of equations:

$$\nabla^2 P_i + k^2_u P_i = 0,\ y > 0$$

$$\nabla^2 P_r + k^2_u P_r = 0,\ y > 0$$

$$\nabla^2 P_t + k^2_l P_t = 0,\ y < 0$$

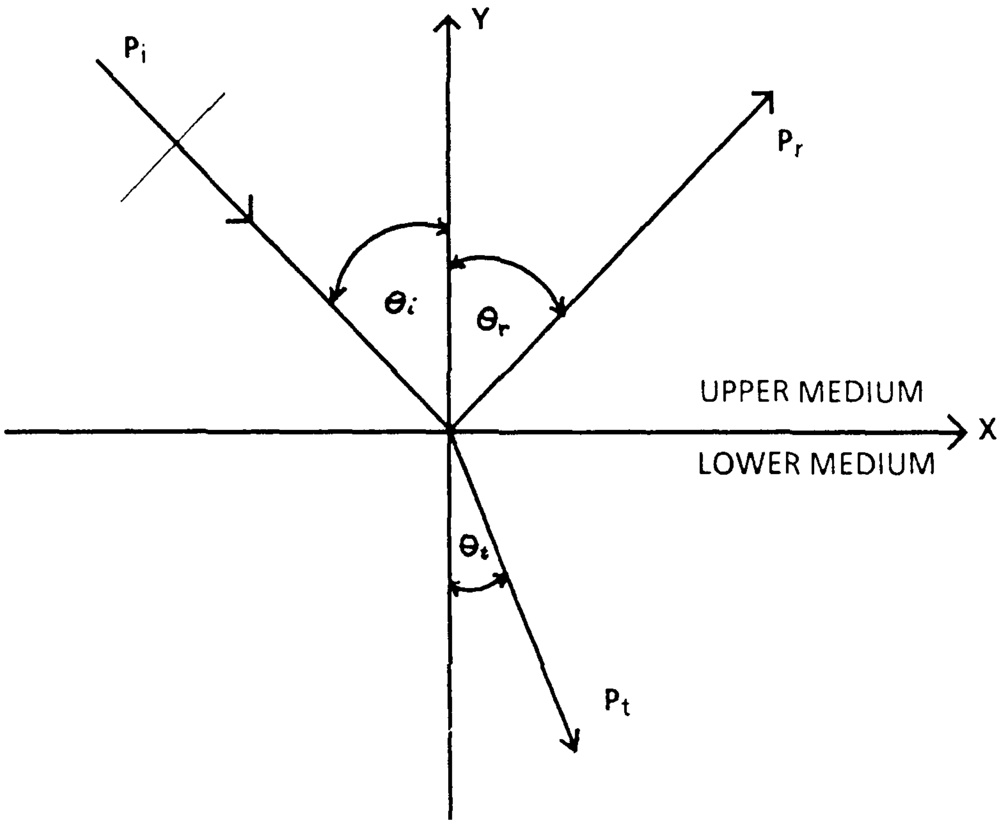

Figure 1.1 Plane Wave Reflection and Refraction

where $k = \omega/c$ = wave number. An $e^{-i\omega t}$ time dependence is assumed throughout the following analysis. Allowing for continuity of pressure at the interface, we have the condition:

$$P_u = P_i + P_r = P_l = P_t$$

or,

$$P_i + P_r = P_t$$

For an ideal fluid, we have:

$$\nabla P = -\rho \nabla \dot{\phi} = i\omega \rho \vec{u}$$

where $\vec{u}$ is the fluid velocity. After taking the dot product of both sides of this equation with $\hat{n}$ we have:

$$\hat{n} \cdot \nabla P = \hat{n} \cdot i\omega \rho \vec{u}$$

where $\hat{n}$ is the unit normal to the interface. This equation yields:

$$\frac{\partial P}{\partial y} = i\omega \rho v,$$

where v is the fluid velocity normal to the interface. Our second boundary condition at the interface equates normal fluid velocities, yielding:

$$\frac{1}{\rho_u} \frac{\partial}{\partial y}(P_i + P_r) = \frac{1}{\rho_l} \frac{\partial}{\partial_y}(P_t),$$

where ρ_u and ρ_l are densities of the two fluid mediums. We define reflected and transmitted plane waves as

$$P_r = Re^{ik_u \vec{x} \vec{d}_r}$$

and

$$P_t = Te^{ik_l \vec{x} \vec{d}_t}.$$

Substituting these into the two interface boundary conditions produces:

$$e^{ik_u x\sin\theta_i}+Re^{ik_u x\sin\theta_r}=Te^{ik_l x\sin\theta_t}$$

$$\frac{k_u}{\rho_u}\left[-\cos\theta_i e^{ik_u x\sin\theta_i}+\cos\theta_r Re^{ik_u x\sin\theta_r}\right]=\frac{-k_l}{\rho_l}\left[\cos\theta_t Te^{ik_l x\sin\theta_t}\right].$$

In order for these two equations to be satisfied for all x, the phases must be equal along the interface. From this we have:

$$ik_u\sin\Theta_i = ik_u\sin\Theta_r = ik_l\sin\Theta_t.$$

This equation yields both the law of refraction and the law governing reflection:

$$\frac{\sin\theta_i}{C_u}=\frac{\sin\theta_t}{C_1}$$

and,

$$\Theta_i = \Theta_r.$$

The solution to the problem for R and T is that

$$R=\frac{\cos\theta_i-\alpha\beta\cos\theta_t}{\cos\theta_i+\alpha\beta\cos\theta_t}$$

$$T=\frac{2\cos\theta_i}{\cos\theta_i+\alpha\beta\cos\theta_t},\ where\ \alpha=\frac{\rho_u}{\rho_l}\ and\ \beta=\frac{C_u}{C_l}.$$

Special cases such as Brewster's angle and the critical angle where total internal reflection occurs can be studied in the context of how R and T vary for differing material properties and indices of reflection.

Reflection and refraction are both important topics in the study of acoustic, electromagnetic, and elastic wave propagation. The following list of books all have good treatments of this subject matter: Halliday and Resnick [Ref. 1], Jones [Ref. 3], Lorrain and Carson [Ref. 4], and Achenbach [Ref. 5].

II. MATHEMATICAL PRELIMINARIES

A. INTRODUCTION

The purpose of this chapter is to present a brief overview of the basic mathematical concepts which are employed in future chapters. The important concepts include the idea of a functional, the simplest variational problem, the Euler-Lagrange necessary condition for the solution of the simplest variational problem, special cases of the Euler-Lagrange equation, an introduction to optimal control, Pontryagin's Maximum Principle, and, the use of Lagrange multipliers. The following references were used in the development of this material: Gelfand and Fomin [Ref. 6], Weinstock [Ref. 2], Leitmann [Ref. 7], Ewing [Ref. 8], Weir [Ref. 9], Bliss [Ref. 10], Pontryagin [Ref. 11], Taylor and Mann [Ref. 12], and, Fleming and Rishel [Ref. 13].

B. FUNCTIONALS

Variable quantities called functionals play an important role in several areas of mathematics, physics and mechanics. A functional is an operator which maps a function onto a real number.

For the problem of wave propagation in a host medium, consider all possible paths joining two given points A and B in the plane. "Suppose that a particle can move along any of these paths, and let the particle have velocity v(x,y) at the point

(x,y). We obtain a functional by associating with each path the time required for the particle to traverse the path." [Ref. 6:p. 1] Theoretical proofs of some results in calculus of variations depend upon concepts from functional analysis. For applications of functional analysis and variational methods in engineering, see Reddy [Ref. 14]. For a discussion on applications of functional analysis to time optimal control problems, consult Hermes and LaSalle [Ref. 15].

C. THE SIMPLEST VARIATIONAL PROBLEM

The simplest variational problem involves the determination of the maxima or minima of a functional of the form:

$$J[y]=\int_a^b F(x,y,y')dx$$

with fixed boundary conditions: y(a) = A and y(b) = B. An example of the simplest problem is finding the shortest plane curve joining two points A and B. That is, the problem is to find the curve, y = y(x), for which the functional

$$J[y]=\int_a^b \sqrt{1+y'^2}\,dx$$

achieves its minimum. The curve in question turns out to be a straight line connecting the two points A and B.

D. THE EULER-LAGRANGE NECESSARY CONDITION

If there exists a continuous, twice-differentiable function that minimizes the simplest problem, what then is the differential equation which satisfies this function? The answer to this question leads to the implementation of the Euler-Lagrange

equation. We include the following sketch of a derivation of the Euler-Lagrange equation for completeness. For an in-depth development of this matter, see Weinstock [Ref. 2:pp. 20-22]. We know that $f(x+h) - f(x) = f'(x)h$ + higher order terms in h. We apply this to our simplest variational problem by considering two functions y and y+h satisfying the same boundary conditions. Let $y(a) = A$ and $(y+h)(a) = A$, then $h(a) = 0$ The same argument holds true for $y(b) = B$. Now, define the following:

$$\Delta J = J[y+h] - J[y]$$

$$\Delta J = \int_a^b [F(x,y+h,y'+h') - F(x,y,y')]dx$$

and use a Taylor series expansion to arrive at:

$\Delta J = \int_a^b (F_y h + F_{y'} h')dx$ + higher order terms. We also define $\delta J[h]$ to be the first variation:

$$\delta J[h] = \int_a^b (F_y h + F_{y'} h')dx \,.$$

Setting the first variation equal to zero, and integrating by parts we obtain:

$$\int_a^b F_y h dx + \int_a^b F_{y'} h' dx = \int_a^b F_y h dx + F_{y'} h \Big|_a^b - \int_a^b h \frac{d}{dx}(F_{y'})dx = 0 \,.$$

Now, since $h(a) = h(b) = 0$,

$$F_{y'} h \Big|_a^b = 0 \,.$$

This leaves us with:

$$\int_a^b \left[F_y - \frac{d}{dx}F_{y'}\right] h dx = 0 .$$

By the fundamental lemma of the calculus of variations, (see Gelfand and Fomin [Ref. 6:p. 9]), it follows that:

if, $\int_a^b \alpha(x)h(x)dx=0$, where h(a) = h(b) = 0, and α and h are C[a,b] then α(x) = 0

for all x ϵ [a,b]. Then,

$$F_y - \frac{d}{d_x}(F_{y'}) = 0 .$$

Thus, the Euler-Lagrange equation for the simplest problem is:

$$F_y - \frac{d}{dx}(F_{y'}) = 0 .$$

Thus, a function y(x) which satisfies the Euler-Lagrange equation and the associated boundary conditions is by *definition the extremizing function.*

The use of the Euler-Lagrange equation does not imply that the solution is an extremum. That is, a solution to the Euler equation presupposes the existence of a maxima or a minima. Just as in the ordinary differential calculus, if we consider $f(x) = x^3$, setting $f'(x) = 3x^2 = 0$ yields neither a minima or a maxima. Another necessary condition is the Legendre necessary condition which states that for the functional:

$$J[y] = \int_a^b F(x,y,y')dx,$$

$y(a) = A$, $y(b) = B$ to have a minimum for the curve $y = y(x)$, the inequality $F_{y'y'} \geq 0$ must be satisfied at every point of the curve. Other necessary conditions are further developed in Leitmann [Ref. 7], and, Bliss [Ref. 10].

E. SPECIAL CASES OF THE EULER-LAGRANGE EQUATION

The Euler-Lagrange equation plays an important role in the calculus of variations and is normally a second-order differential equation. There are some special cases for which Euler's equation can be reduced to a first-order differential equation. For instance, if the functional has no y dependence Euler's equation becomes $F_{y'} = C$. If the functional has no x dependence then the corresponding Euler equation is $F - y'F_{y'} = C$. Finally, if the functional does not depend upon y', Euler's equation reduces to $F_y(x,y) = 0$ and therefore is not a differential equation. For further treatment of this subject, see Gelfand and Fomin [Ref. 6:pp. 18-19], and, Weinstock [Ref. 2:p. 52].

F. OPTIMAL CONTROL

The problem of optimal control is intimately related to certain problems in the calculus of variations. The following is an outline of this process and is taken from Gelfand and Fomin [Ref. 6:pp. 218-219]. Suppose the state of a physical system is characterized by n real numbers $x^1,\ldots,x^n$, forming a vector $x = (x^1,\ldots,x^n)$ in the

n-dimensional phase space X of the system. Further, suppose the state varies with time in the way described by the system of differential equations:

$\frac{dx^i}{dt} = f^i(x^1,\dots,x^n,u^1,\dots,u^k)(i=1,\dots,n)$. Here the k real numbers $u^1,\dots,u^k$ form a vector $u = (u^1,\dots,u^k)$ belonging to some fixed control region Ω, which we take to be a subset of k-dimensional Euclidean space. The $f^i(x,u)$ are n continuous functions defined for all $x \in X$ and all $u \in \Omega$. Now suppose we specify a vector function u(t), $t_0 \le t \le t_1$ called the control function, with values in Ω. Then, substituting u = u(t) in the differential equations above yields:

$$\frac{dx^i}{dt} = f^i[x^1,\dots,x^n,u^1(t),\dots,u^k(t)](i=1,\dots,n).$$

For every initial value of $x_0 = x(t_o)$, this system has a definite solution, called a trajectory. The aggregate, $U = \{u(t),t_0,t_1,x_0\}$, consisting of a control function u(t), an interval $[t_0,t_1]$ and an initial value $x_0 = x(t_0)$, will be called a control process. Thus, to every control process, there corresponds a trajectory. Now let $f^0(x^1,\dots,x^n,u^1,\dots,u^k)$ be a function which is defined, together with its partial derivatives, $\frac{\partial f^o}{\partial x^i}(i=1,\dots,n)$,

for all $x \in X$, and all $u \in \Omega$.

To every control process U, we assign the number $\int_{t_o}^{t_1} f^o(x,u)dt$. Then, the control process is said to be optimal if the inequality, $J[U] \le J[U^*]$ holds for any

other control process U* carrying the given point x_0 into the point x_1. By the optimal trajectory, we mean the one corresponding to the optimal control process. The following books have detailed explanations of this material and are recommended for further study: Pontryagin [Ref. 11], Leitmann [Ref. 7:pp. 79-83], and Berkovitz [Ref. 16].

G. PONTRYAGIN'S MAXIMUM PRINCIPLE

We now state the main theorem of optimal control theory, the maximum principle. For further reading on this theorem and its proof, see Pontryagin [Ref. 11:p. 11], Gelfand and Fomin [Ref. 6:p. 222], and Leitmann [Ref. 7:p. 118].

Let $U = \{u(t), t_0, t_1, x_0\}$ be an admissible control process, and let x(t) be the corresponding integral curve of the system,

$\frac{dx^i}{dt} = f^i(x,u)$ $(i = 0,1,\ldots,n)$, passing through the point $(0, x_0^1, \ldots, x_0^n)$ for $t = 0$, and

satisfying the conditions: $x^1(t_1) = x_1^1, \ldots, x^n(t_1) = x_1^n$ for $t = t_1$. Then if the control process U is optimal there exists a continuous vector function $\Psi(t) = (\Psi_0(t), \Psi_1(t), \ldots, \Psi_r(t))$ such that:

1. The function $\Psi(t)$ satisfies the system, $\frac{d\Psi_i}{dt} = -\sum_{\alpha=0}^{n} \frac{\partial f^\alpha(x,u)}{\partial x^i} \Psi_\alpha$ $(i = 0,1,\ldots,n)$, for $x = x(t)$, $u = u(t)$.

2. For all t in $[t_0, t_1]$, the function $H(\Psi,x,u)=\sum_{\alpha=0}^{n} \Psi_\alpha f^\alpha(x,u)$ achieves its maximum for $u = u(t)$.

3. The relations, $\psi_0(t_1) \leq 0$, $G[\psi(t_1),u(t_1)] = 0$ hold at the time t_1, where $G(\psi,x) = \underset{u \in \Omega}{Sup}\, H(\Psi,x,u)$.

H. LAGRANGE MULTIPLIERS

"A necessary condition for a minimum (or maximum) of $F(x,y,...,z)$ with respect to variables $x,y,...,z$ that satisfy $G_i(x,y,...,z) = C_i$, $(i = 1,2,...,N)$, where C_i are given constants is:

$$\frac{\partial F^*}{\partial x}=\frac{\partial F^*}{\partial y}=\ldots=\frac{\partial F^*}{\partial z}=0,$$

where $F^*=F+\sum_{C=1}^{N} \lambda_i G_i$. The constants $\lambda_1, \lambda_2, \ldots, \lambda_N$, introduced as undetermined

Lagrange multipliers, are evaluated together with the minimizing (or maximizing) values of $x,y,...,z$ by means of the set of equations consisting of those listed above." [Ref. 2:p. 6].

Taylor and Mann [Ref. 12], and Weir [Ref. 9] give good treatments of the Lagrange multiplier method with several examples.

III. REFLECTION

A. INTRODUCTION

Mathematical problems are frequently inspired through observations of various physical phenomena. In fact several solutions to mathematical problems are inspired by nature. The physics of the problem provide us the clues, without which we would have very little chance of succeeding in finding solutions. One of those problems suggested by nature involves the reflection of light particles on a plane mirror.

We know from Fermat's Principle that a light particle will choose the path which will get it to its destination in the shortest amount of time. We also know that the shortest distance between two points is a straight line. But what happens to these light particles when they undergo a reflection? Do they still choose the shortest path and if so, what is the shortest path? This chapter answers these questions and others. The material in this chapter, although well known, is presented for completeness. We review the classical results of reflection and complete our discussion with a derivation of the reflection principle using the method of Lagrange multipliers. The following references were used extensively in the preparation of this chapter: Courant and Robbins [Ref. 17], Polya [Ref. 18], Kazarinoff [Ref. 19], Williams [Ref. 20], and Chakerian and Ghandehari [Ref. 21].

B. HERON'S PRINCIPLE

Our problem, as shown in Figure 3.1 is as follows: given two points A and B, both on the same side of a straight line L, all in the same plane, what is the point C on L which minimizes the sum of the distances from A and B?

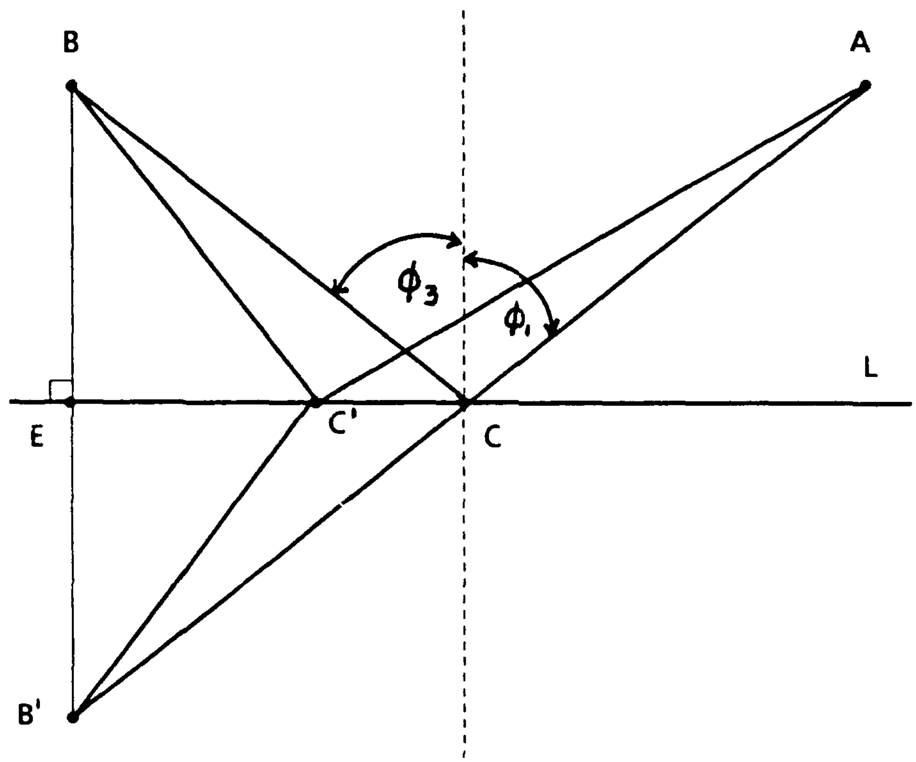

Figure 3.1 Heron's Problem

That is, find the point C such that: AC + BC is a minimum. If we can show that:

$$AC' + BC' > AC + BC$$

then we prove that the point C does in fact minimize the sum of the distances from A and B to C. We know that:

$$AC' + BC' = AC' + B'C'$$

for B′ the point reflected in the line L through the perpendicular to L at E. This follows since the triangles BEC′ and B′EC′ are congruent triangles and the line L is the perpendicular bisector of the line segment BB′. Now by the triangle inequality, we have:

$$AC' + B'C' > AB'$$

Looking at Figure 3.1, we can see that:

$$AB' = AC + CB'$$

Now by the same argument used above (congruent triangles), we know that:

$$CB' = CB.$$

Then it follows that :

$$AB' = AC + CB.$$

Therefore,

$$AC' + BC' > AC + BC$$

as we desired. So we have in fact established that the point C does minimize the sum of the distances from A and B. It follows from geometry that the angles Φ_1 and Φ_3 are equal and thus we have the law which governs the principle of reflection:

$$\Phi_1 = \Phi_3$$

Therefore, the reflected ray of light takes the shortest possible course between the two points A and B. This reflected ray stays within the plane of incidence since

any other possible direction of reflection would not produce the shortest possible path from the point A to the point B. This discovery is due to Heron, the Alexandrian scientist of the first century A.D. [Ref. 17]. "Simple as it is, Heron's discovery deserves a place in the history of science. It is the first example of the use of a minimum principle describing a physical phenomenon." [Ref. 18]

Heron's reflection principle can be generalized in Minkowski spaces. A Minkowski space is simply a finite dimensional normed linear space. A generalization of Heron's reflection principle using Lagrange multipliers in a Minkowski plane and related applications to conics is given in Ghandehari [Ref. 22]. For a generalization of Steiner's problem (Fermat's problem) see Chakerian and Ghandehari [Ref. 21].

Consider the following problem: given an area A and one side of a triangle, c, which triangle is the one for which the sum of the other sides a and b is the smallest? See Figure 3.2. [Ref. 17:p. 332]

Prescribing the side c and the area A of a triangle is equivalent to prescribing the side c and the height h on c, since $A = \frac{1}{2}(ch)$. Therefore, our problem is reduced to finding the point R such that the distance from R to the line PQ is equal to the height, h and such that the sum $a + b$ is a minimum. From the first condition, it follows that R must lie on a line parallel to the line PQ at a distance h. This R is given by Heron's Theorem for the special case where P and Q are equally distant from L.

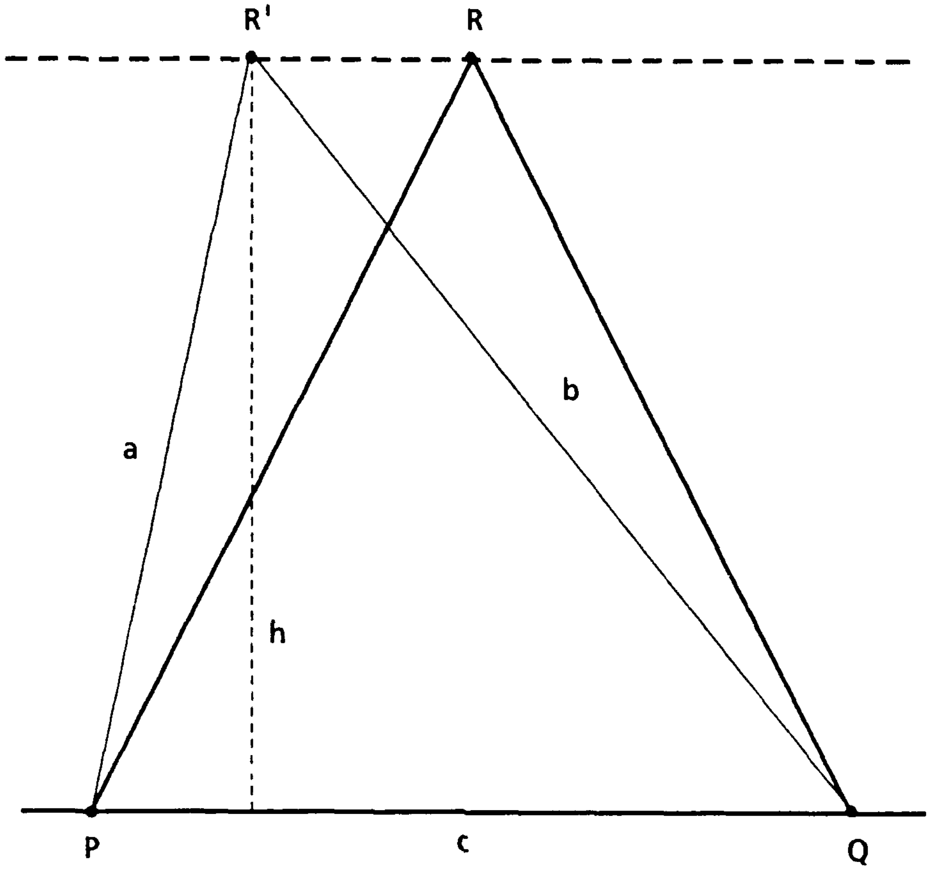

Figure 3.2 Application of Heron's Principle

C. STEINER'S PROBLEM

The following example comes from Polya [Ref. 18:p. 145], and illustrates the use of the reflection principle. It is commonly known as Steiner's problem named after Jacob Steiner, the famous representative of geometry at the University of Berlin in the early nineteenth century. Three towns intend to construct three roads to a common traffic center which should be chosen so that the total cost of the road

construction is minimized, assuming cost is proportional to length alone. Simply put, given three points, find a fourth such that the sum of the distances from the three points to the fourth is minimized. See Figure 3.3. Let A, B, and C denote the three towns and X the fourth point in question. Also, assume that all interior angles a, b, and c are all less than 120°.

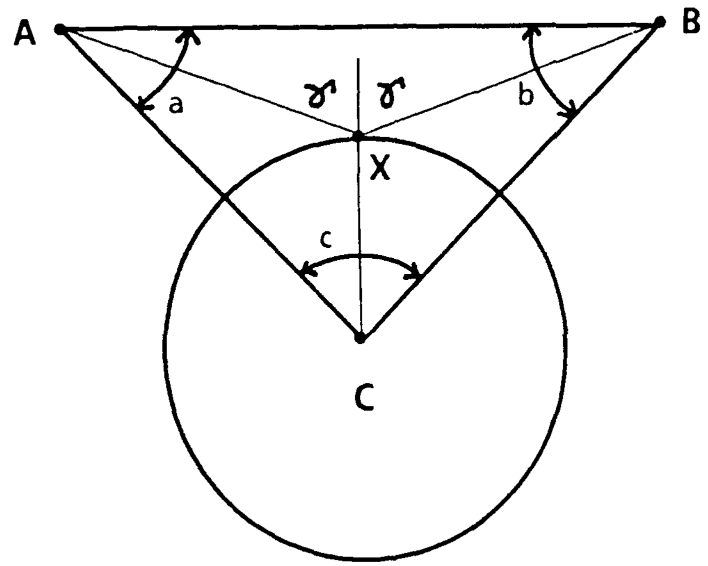

Figure 3.3 Steiner's Problem

We want to minimize the following expression: C = AX + BX + CX. If we fix one of these distances, say CX, then we have a problem which resembles Heron's problem. Therefore, we let CX = r and minimize AX + BX. In this case, we are dealing with reflection in a circular mirror. We know that light particles move in such a way that the angle of incidence is equal to the angle of reflection. So in our problem the angle <AXB must be bisected by a straight line passing through C and X. Similarly the angles <AXC and <BCX must also be bisected by lines through B and A respectively. As such, the three straight lines joining X to A , B, and C dissect the plane into six angles, the common vertex being X. It then follows that all six angles are equal and equal to 60°. Therefore the three roads diverging from the traffic center do so at 120° angles. If one of the interior angles is at least 120°, then the common vertex which minimizes the sum of the distances from the three points A, B, C will be that vertex which has an interior angle of at least 120°.

D. REFLECTION PROPERTIES OF CONICS

1. Ellipse

The reflection principle applied to an ellipse indicates that the focal radii make equal angles with the tangent line. The following derivation is taken from Kazarinoff [Ref. 19]. Let L be a tangent to an ellipse with foci F_1 and F_2, see Figure 3.4. Let P be the point of tangency, and let Q be any other point on L. Since Q is outside of the ellipse:

$$QF_1 + QF_2 > PF_1 + PF_2.$$

Therefore, F_1PF_2 is the shortest path from F_1 to L to F_2 and the angles α and ß are the same.

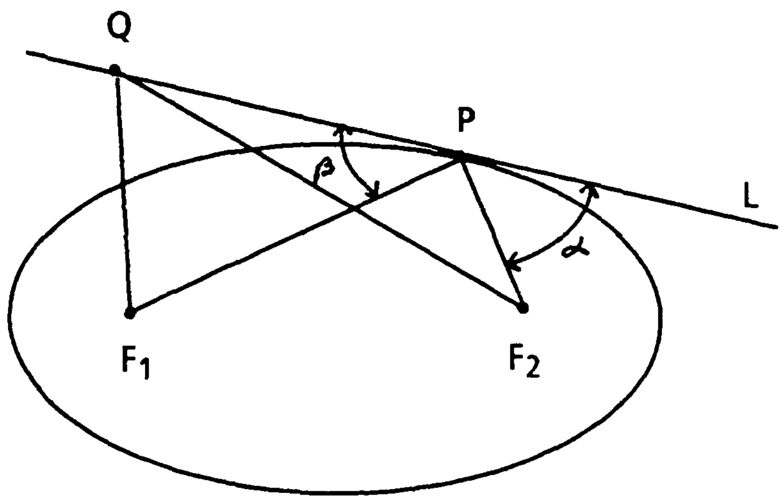

Figure 3.4 Reflection Properties of an Ellipse

2. Parabola

An important property of parabolas is that any ray of light emanating from the focus of a parabolic mirror is always reflected along a line parallel to the axis of the parabola. This property is equivalent to saying that the angles α and ß, see Figure 3.5, formed by lines through the focus F and an arbitrary point P on the parabola and through P parallel to the axis of the parabola are equal. For a proof of this property, refer to Williams [Ref. 20].

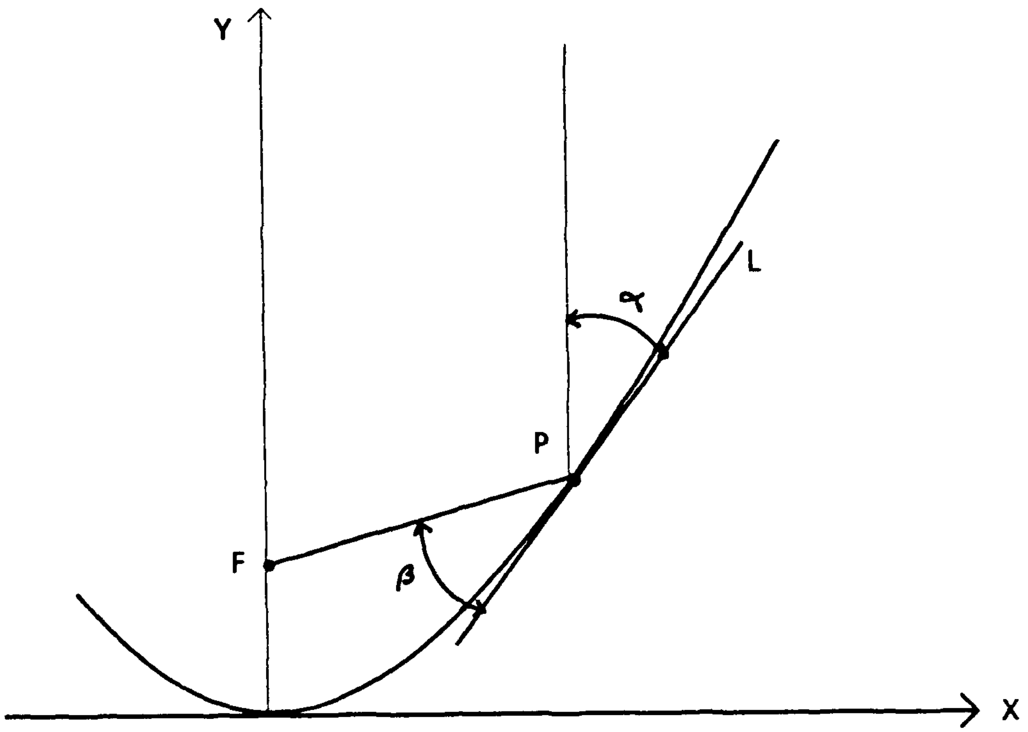

Figure 3.5 Reflection Properties of a Parabola

3. Hyperbola

The tangent properties of the ellipse are related to those of the hyperbola. Figure 3.6 depicts a hyperbola with P and Q as its foci. The tangent line L passing through an arbitrary point R on the hyperbola bisects the angle subtended at that point by the foci of the hyperbola. [Ref. 17:p. 335] Therefore, the angles α and ß are the same.

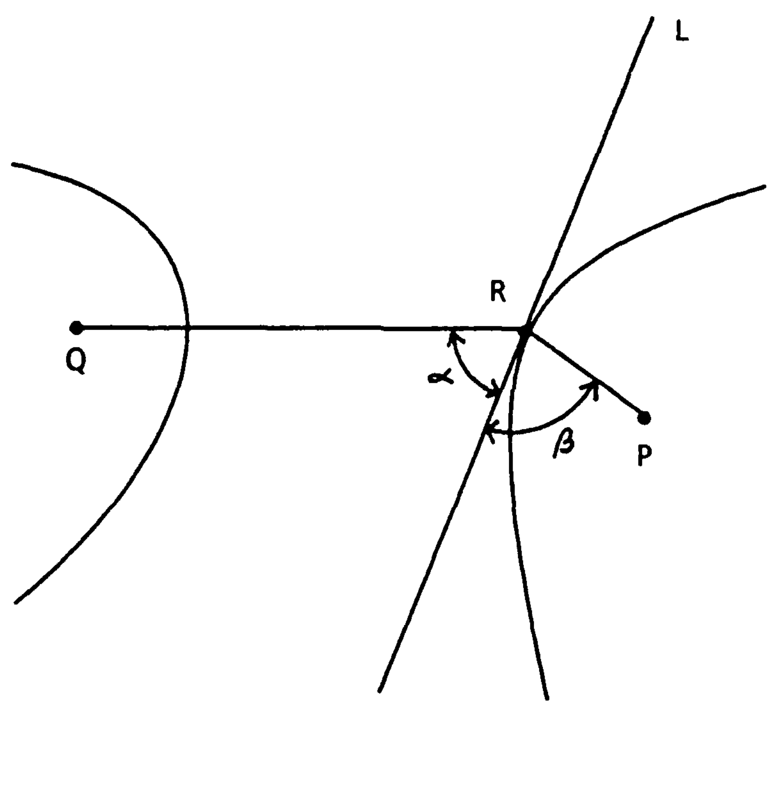

Figure 3.6 Reflection Properties of a Hyperbola

E. A GENERALIZATION OF REFLECTION

We now turn our attention to a derivation of the law governing the principle of reflection involving a ray of light passing through a homogeneous medium, see Figure 3.7. We make use of the method of Lagrange multipliers in developing our solution.

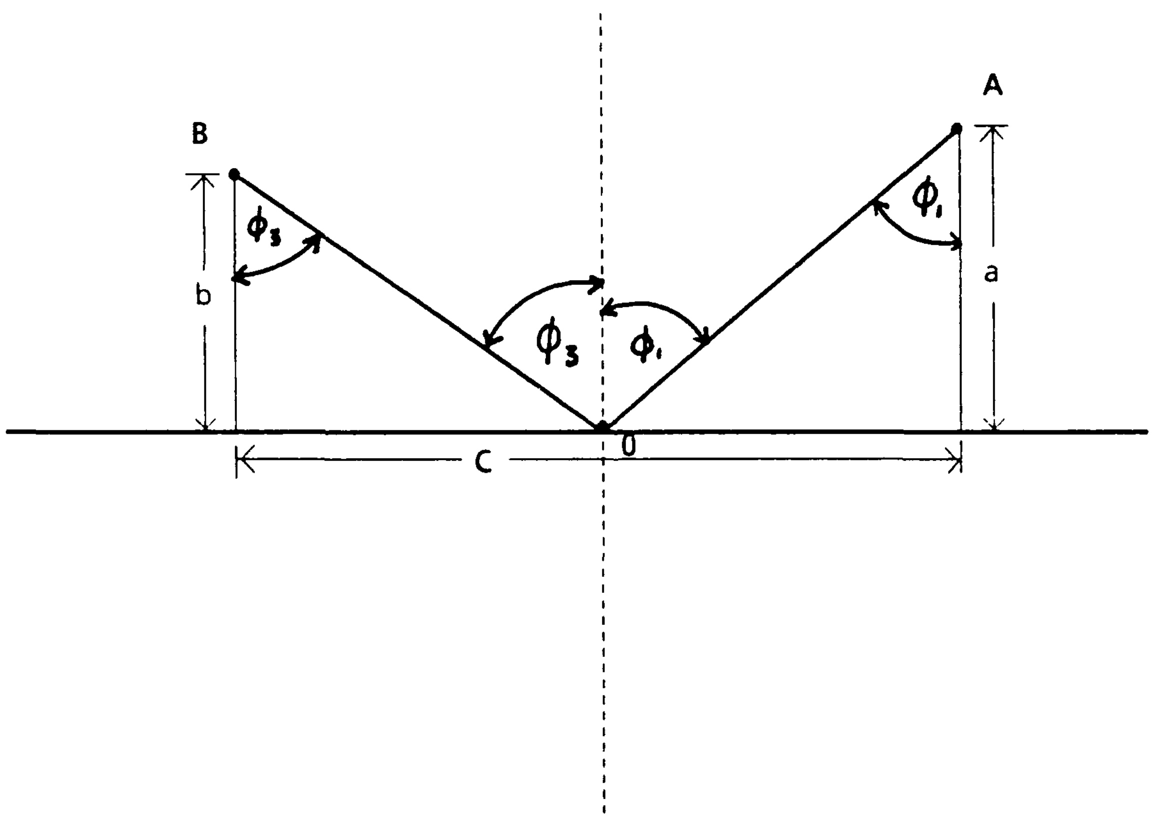

Figure 3.7 Reflection in a Homogeneous Isotropic Medium

We know that:

$$\cos\phi_1 = \frac{a}{OA}$$

$$OA = \frac{a}{\cos\phi_1} = a\sec\phi_1$$

Similarly, for Φ_3:

$$OB = b\sec(\Phi_3)$$

Applying Fermat's Principle, we wish to minimize the following expression:

$a\sec(\Phi_1) + b\sec(\Phi_3)$, subject to the constraint that:

$$a\tan(\Phi_1) + b\tan(\Phi_3) = c.$$

Now construct a new function, call it h such that:

$$h(\phi_1,\phi_3,\lambda) = a\sec(\phi_1) + b\sec(\phi_3) - \lambda[a\tan(\phi_1) + b\tan(\phi_3) - c]$$

Applying the method of Lagrange multipliers [Ref. 9], we set:

$$\frac{\partial h}{\partial \phi_1} = \frac{\partial h}{\partial \phi_3} = \frac{\partial h}{\partial \lambda} = 0,$$

which yields the following system of equations:

$$\frac{\partial h}{\partial \phi_1} = a\sec\phi_1\tan\phi_1 - \lambda a\sec^2\phi_1 = 0$$

$$\frac{\partial h}{\partial \phi_3} = b\sec\phi_3\tan\phi_3 - \lambda b\sec^2\phi_3 = 0$$

$$\frac{\partial h}{\partial \lambda} = -a\tan\phi_1 - b\tan\phi_3 - c = 0$$

which yields:

$$\frac{a\sin\phi_1}{\cos^2\phi_1} = \frac{\lambda a}{\cos^2\phi_1}$$

$$\frac{b\sin\phi_3}{\cos^2\phi_3} = \frac{\lambda b}{\cos^2\phi_3}$$

$$-a\tan\phi_1 - b\tan\phi_3 - c = 0$$

We then have:

$$\sin\phi_1 = \lambda \ \textit{and} \ \sin\phi_3 = \lambda$$

which yields the law governing the principle of reflection:

$$\phi_1 = \phi_3.$$

IV. REFRACTION

A. INTRODUCTION

"A stick bobbing in the water looks sharply bent. We reason that the light that follows a straight course in the water as in the air undergoes an abrupt change of direction in emerging from the water into the air." [Ref. 18] This is the phenomenon of refraction. The light that travels with a known velocity through air, travels with a different velocity through water. Such a difference in velocity explains in part the phenomenon of refraction. This chapter will deal explicitly with Snell's Law of Refraction and is the focus of this paper. We develop several derivations, the most important of which is one in which the velocity of light depends upon the direction the light takes in each medium. We use the method of Lagrange multipliers and optimal control theory to derive this generalization. The following references were useful in the preparation of this chapter, Apostol [Ref. 23], Polya [Ref. 18], Weinstock [Ref. 2], Taylor and Mann [Ref. 12], Pontryagin [Ref. 11], Gelfand and Fomin [Ref. 6], and, Halliday and Resnick [Ref. 1].

B. A SIMPLE EXAMPLE

Our first development, as an introduction, comes from A. V. Baez, which was published by Apostol [Ref. 23].

It is known that of all possible paths connecting two points A and B in space, a light ray leaving A chooses the path which enables it to reach B in the least time. This is Fermat's Principle from which the law of refraction and the reflection principle follow.

Consider the following problem: A man is in a boat at a point P, one mile from the nearest point A on the shore. He wishes to go to a point B which is farther down the shore, M miles from A. He also knows that he can walk faster than he can row the boat. If he can row r miles an hour and walk w miles an hour, toward what point C should he row in order to reach B in the least amount of time? See Figure 4.1. Looking at Figure 4.1 if we let AC = x, Snell's Law of Refraction indicates that:

$$\frac{\sin\phi_1}{\sin\phi_2} = \frac{r}{w}$$

Let $\Phi_2 = 90^\circ$ and use geometry to obtain:

$$\frac{x}{\sqrt{1+x^2}} = \frac{r}{w}$$

Solving for x we have our solution:

$$x = \frac{r}{\sqrt{w^2-r^2}}$$

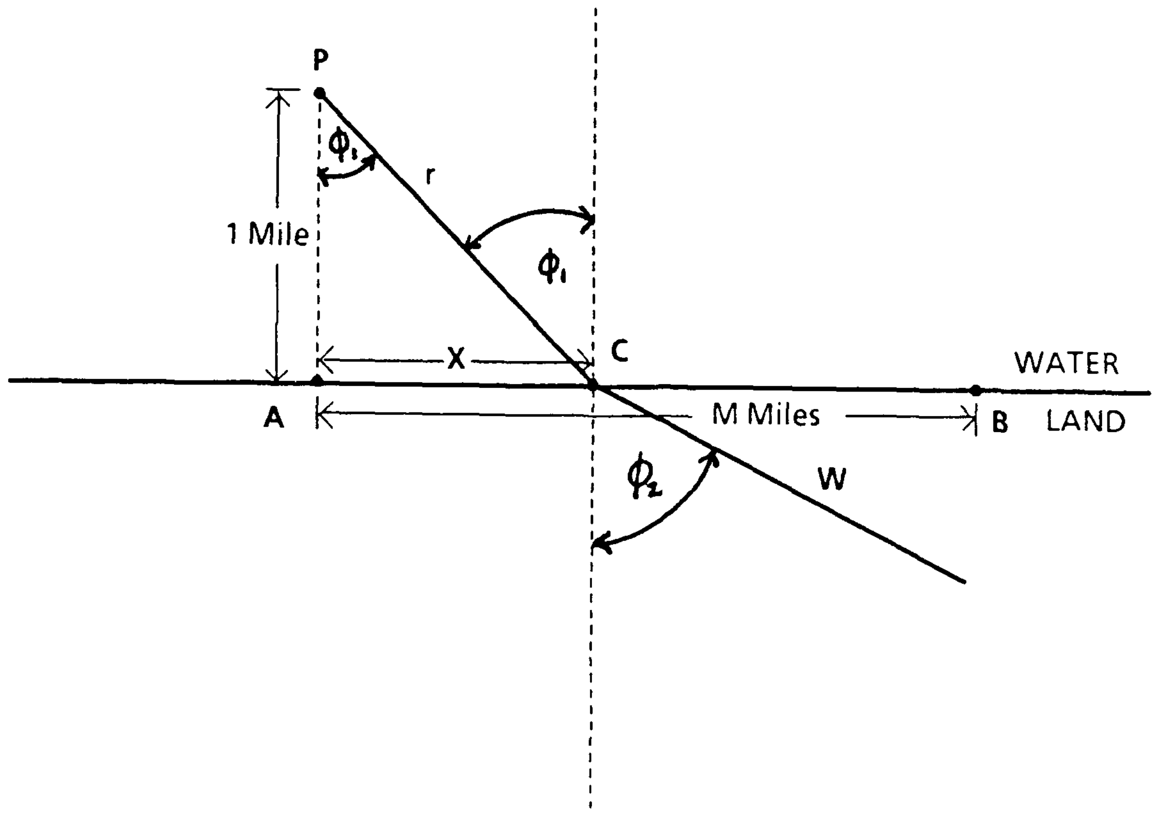

Figure 4.1 Application of Law of Refraction

This example serves to illustrate the fact that we can analyze and solve certain problems avoiding the use of calculus by simply applying Snell's Law of Refraction.

C. GENERALIZATIONS OF SNELL'S LAW OF REFRACTION

1. Optically Isotropic Media

Our first derivation of Snell's Law of Refraction involves the passage of light between two points in two optically isotropic media. See Figure 4.2. That is, the velocities of the particles in both media are constant but different [Ref. 2]. Consider for example the path of a light particle emanating from a source at (X_1,Y_1) in water through a thin plate of glass, to a point (X_2,Y_2) also in wat^r.

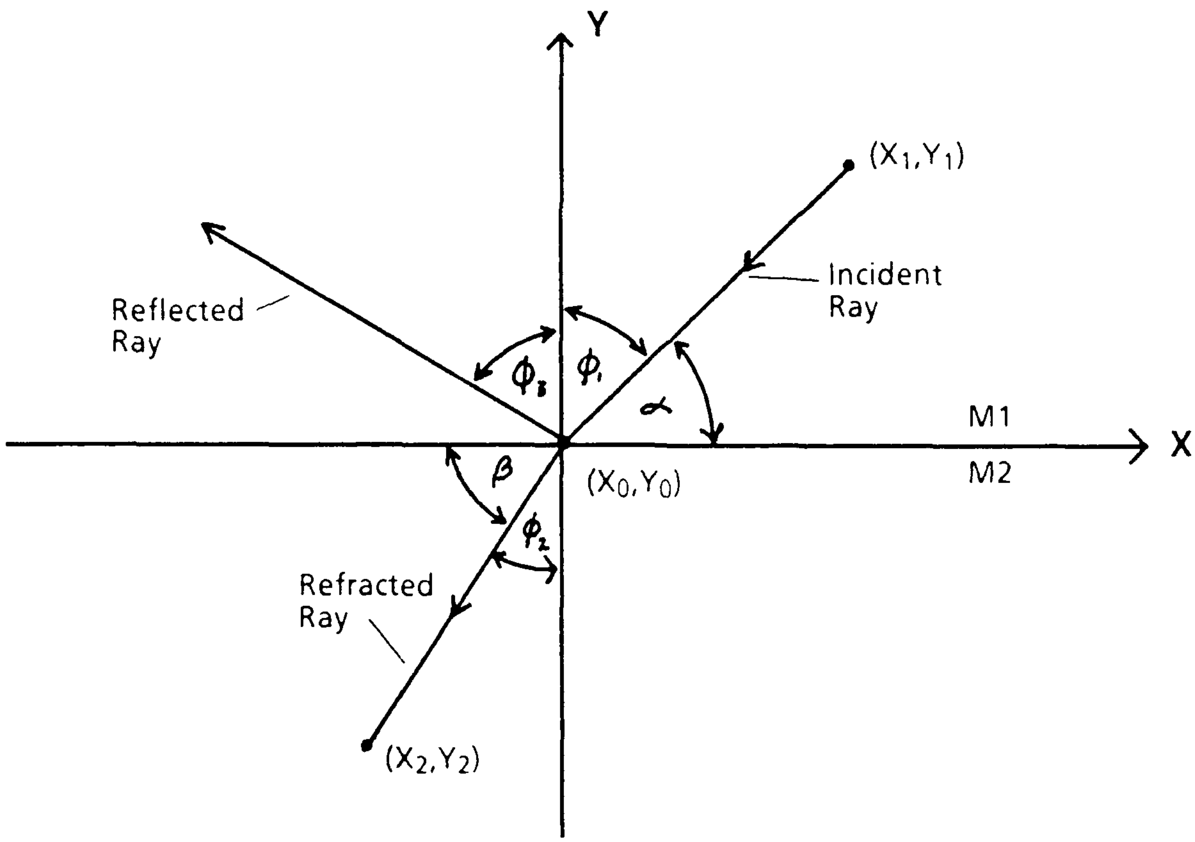

Figure 4.2 Refracted, Reflected, and Incident Rays

Applying Fermat's Principle from the point (X_1,Y_1) in medium 1 to a point (X_2,Y_2) in medium 2, with respective light velocities of v_1 and v_2 we arrive at:

$$T=\frac{\sqrt{(X_1-X_0)^2+(Y_1-Y_0)^2}}{v_1}+\frac{\sqrt{(X_0-X_2)^2+(Y_0-Y_2)^2}}{v_2}$$

where T = time = arc length/velocity = $\sum \frac{ds}{\frac{ds}{dt}}=\sum dt$

Again Fermat's Principle claims that the light particle will take the path of shortest time between two points or,

$$\frac{dT}{dX_0}=\frac{X_0-X_2}{v_2\sqrt{(X_0-X_2)^2+(Y_0-Y_2)^2}}-\frac{X_1-X_0}{v_1\sqrt{(X_1-X_0)^2+(Y_1-Y_0)^2}}=0$$

$$Since\ \sin\phi_1=\frac{X_1-X_0}{\sqrt{(X_1-X_0)^2+(Y_1-y_0)^2}},and\ \sin\phi_2=\frac{X_0-X_2}{\sqrt{(X_0-X_2)^2+(Y_0-Y_2)^2}}$$

Then the expression becomes:

$$\frac{\sin\phi_1}{v_1}=\frac{\sin\phi_2}{v_2}$$

2. Optically Isotropic Media Using Lagrange Multipliers

We now attempt to solve the same problem, that is the propagation of light in two isotropic media using the method of Lagrange multipliers [Ref. 12]. See Figure 4.3.

Looking at Figure 4.3 it is easy to see that $\cos(\phi_1)=\frac{a}{OA}$ or,

OA=asec (Φ_1). Now applying Fermat's Principle we want to sum the arc lengths, divided by their respective velocities, to arrive at an expression for time:

$$T=\frac{a\sec\phi_1}{v_1}+\frac{b\sec\phi_2}{v_2}$$

subject to the constraint that:

$$a\tan(\Phi_1) + b\tan(\Phi_2) - C = 0$$

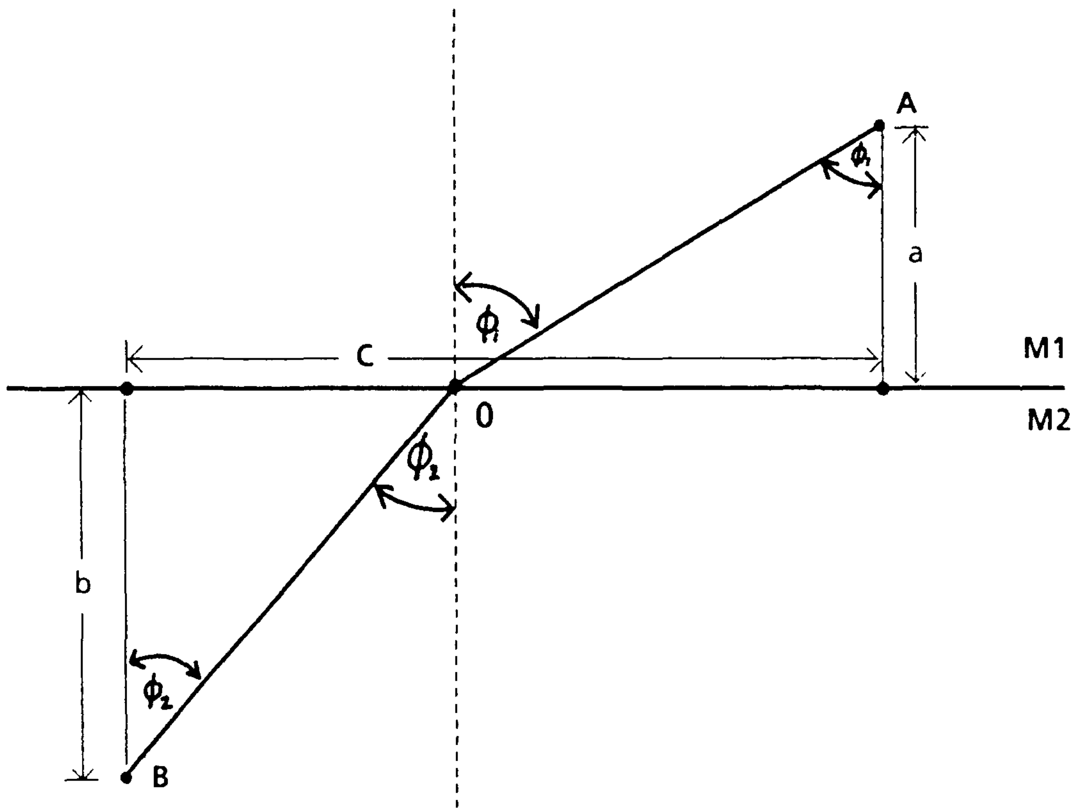

Figure 4.3 Propagation of Light in Two Isotropic Homogeneous Media

Call the constraining function $g(\Phi_1,\Phi_2)$, that is let:

$$g(\Phi_1,\Phi_2) = a\tan(\Phi_1) + b\tan(\Phi_2) - C = 0$$

By the method of Lagrange multipliers [Ref. 9], we form a new function h where:

$h = T - \lambda g(\Phi_1,\Phi_2)$

Setting $\frac{\partial h}{\partial \phi_1} = \frac{\partial h}{\partial \phi_2} = \frac{\partial h}{\partial \lambda} = 0$ yields the following three equations:

$$\frac{a}{v_1}\sec\phi_1\tan\phi_1 - \lambda a\sec^2\phi_1 = 0$$

$$\frac{b}{v_2}\sec\phi_2\tan\phi_2 - \lambda b\sec^2\phi_2 = 0$$

$$a\tan\Phi_1 - b\tan\Phi_2 - C = 0$$

which yield:

$$\frac{a\sin\phi_1}{v_1\cos^2\phi_1} = \frac{\lambda a}{\cos^2\phi_1}$$

$$\frac{b\sin\phi_2}{v_2\cos^2\phi_2} = \frac{\lambda b}{\cos^2\phi_2}$$

$$a\tan\Phi_1 - b\tan\Phi_2 - C = 0$$

We then have:

$$\frac{\sin\phi_1}{v_1} = \lambda \ \ and \ \ \frac{\sin\phi_2}{v_2} = \lambda$$

which again yields Snell's Law:

$$\frac{\sin\phi_1}{v_1} = \frac{\sin\phi_2}{v_2}$$

3. Optically Anisotropic Media With Velocity a Function of Depth

Now, with the help of the calculus of variations we consider the case where the velocity is not constant but rather a function of depth. In other words, consider the propagation of light in two optically anisotropic media where the velocity of the light particles is some function of y [Ref. 24] and [Ref. 26]. Again, using Fermat's Principle, we arrive at the following equation which we want to minimize:

$$T\int_{x_1}^{x_2} \frac{ds}{\frac{ds}{dt}} = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{v(y)} dx$$

This functional is of the form $F(y,y')$, therefore, it represents a special case of the Euler-Lagrange equation when the integrand has no x dependence, [Ref. 6] and [Ref. 11]. Starting with the basic Euler-Lagrange equation:

$$F_y - \frac{d}{dx}(F'_y) = 0$$

We derive an expression which will simplify this second order differential equation into a first order differential equation:

$$F_y - \frac{d}{dx}(F'_y) = F_y - F_{y'y}y' - F_{y'y'}y''$$

Then multiplying by y':

$$F_y y' - F_{y'y} y'^2 - F_{y'y'} y' y'' = \frac{d}{dX}[F - y'F_{y'}]$$

Recalling that a first integral of a system of differential equations is a function which has constant value along the solution curves of the system, we can then say that the first integral of Euler's equation is:

$$F - y'Fy' = C$$

Using this equation, we now solve the differential equation:

$$\frac{\sqrt{1+y'^2}}{v(y)} - \frac{y'y'}{\sqrt{1+y'^2}\,(v(y))} = C$$

From Figure 4.2, $y' = \tan\alpha$, *where* $\phi_1 = \frac{\pi}{2} - \alpha$. Hence,

$$\frac{\sec^2\alpha}{\sec\alpha(v(y))} - \frac{\tan^2\alpha}{\sec\alpha(v(y))} = C$$

or,

$$\frac{\cos\alpha}{v} = C$$

therefore,

$$\frac{\cos(\frac{\pi}{2} - \phi_1)}{v} = C$$

from which we arrive at our familiar expression of Snell's Law, mainly that:

$$\frac{\sin(\phi_1)}{v_1} = \frac{\sin(\phi_2)}{v_2}$$

4. Optically Anisotropic Media With Velocity a Function of Direction

Consider the propagation of light in two anisotropic media where the velocity of the light particles is some function of direction. That is $v_1 = v_1(\alpha)$ and $v_2 = v_2(\beta)$, where α and β are the angles formed by the incident ray and the horizontal, and the refracted ray and the horizontal, respectively, see Figure 4.4. We solve this problem using variational methods and show that the result is:

$$\frac{1}{v_1^2(\alpha)} \frac{d}{d\alpha}[v_1(\alpha)\sin(\alpha)] = \frac{1}{v_2^2(\beta)} \frac{d}{d\beta}[v_2(\beta)\sin(\beta)] = C$$

which, for the simple case of constant velocities (homogeneous media) reduces easily to our standard form of Snell's Law:

$$\frac{\sin\phi_1}{v_1} = \frac{\sin\phi_2}{v_2}$$

Let $y' = \tan(\alpha)$, then $\alpha = \tan^{-1}(y')$. Applying Fermat's Principle, we wish to minimize:

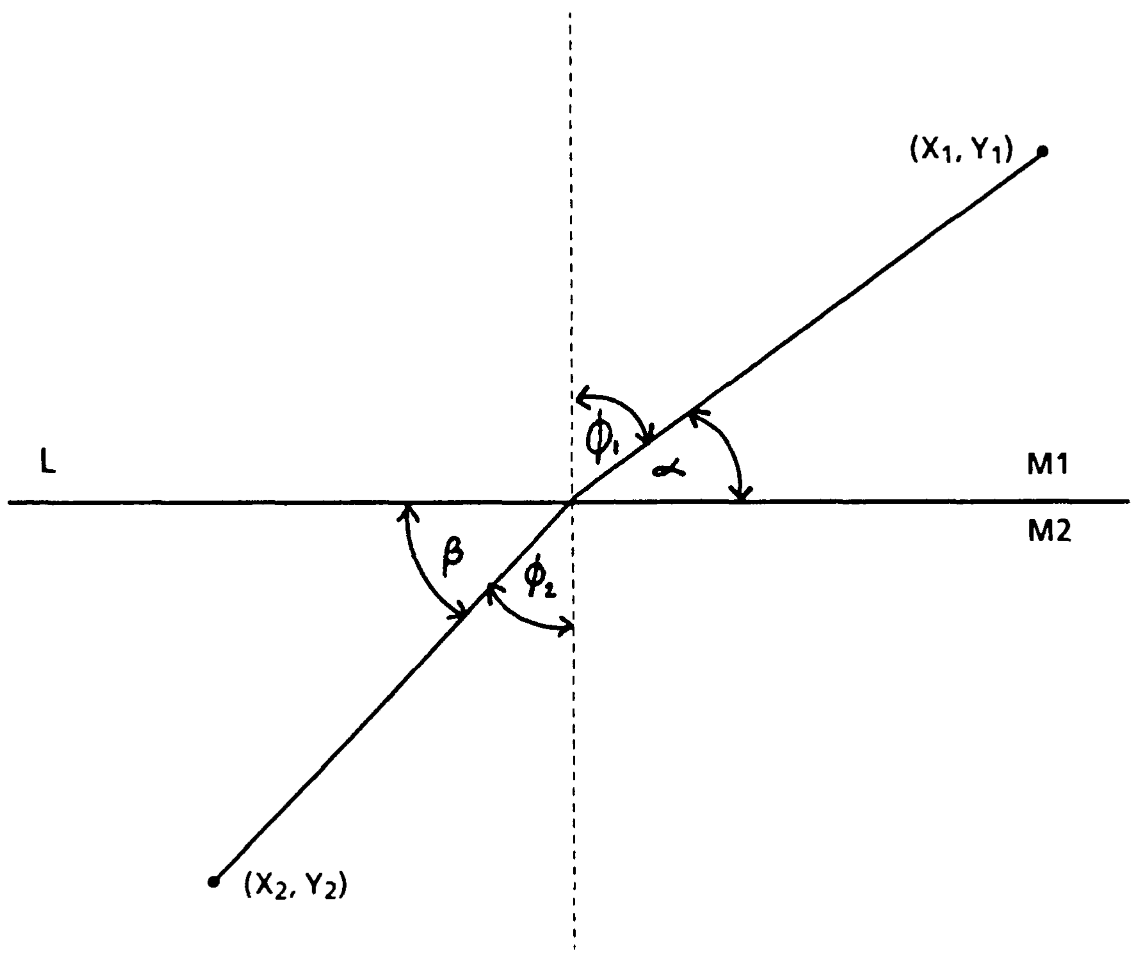

Figure 4.4 Propagation of Light in Two Anisotropic Media

$$T = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{v_1 \tan^{-1}(y')} dx .$$

Let $\dfrac{1}{v_1(\tan^{-1}(y'))} = f(y')$. This is done to simplify the arithmetic involved. Later, we

replace f(y′) by $\dfrac{1}{v_1(\tan^{-1}(y'))}$.

Our problem is now to minimize:

$$T = \int_{x_1}^{x_2} \sqrt{1+y'^2} f(y') dx .$$

This functional is of the form F(y,y′), therefore we again make use of the special case of the Euler - Lagrange equation, [Ref. 6] and [Ref. 11].

Euler's equation then is:

$$F - y'F_{y'} = C \text{ or,}$$

$$f(y')\sqrt{1+y'^2} - y'[f'(y')\sqrt{1+y'^2} + \frac{y'}{\sqrt{1+y'^2}}(f(y'))] = C$$

which further reduces to:

$$\frac{f(y')}{\sqrt{1+y'^2}} - y'f'(y')\sqrt{1+y'^2} = C .$$

This by itself is a more general expression than that obtained by the method of Lagrange multipliers [Ref. 26]. Replacing our earlier definition of f(y′) and y′ we now have:

$$\frac{1}{v_1(\alpha)\sqrt{1+\tan^2\alpha}} - \tan\alpha\sqrt{1+\tan^2\alpha}\left(\frac{d}{dy'}\left(\frac{1}{v_1(\alpha)}\right)\right) = C$$

$$\frac{\cos\alpha}{v_1(\alpha)} - \tan\alpha\sec\alpha\left[\frac{\frac{-1}{1+\tan^2\alpha}v_1'(\alpha)}{v_1^2(\alpha)}\right] = C$$

or,

$$\frac{\cos\alpha}{v_1(\alpha)} - \frac{\sin\alpha}{\cos^2\alpha}\left[\frac{-\cos^2\alpha(v_1'(\alpha))}{v_1^2(\alpha)}\right] = C.$$

Now establishing a common denominator of $v_1^2(\alpha)$:

$$\frac{v_1(\alpha)\cos\alpha + \sin\alpha\, v_1'(\alpha)}{v_1^2(\alpha)} = C$$

from which it can be seen that:

$$\frac{1}{v_1^2(\alpha)}\frac{d}{d\alpha}\left[v_1(\alpha)\sin\alpha\right] = C$$

or,

$$\frac{1}{v_1^2(\alpha)}\frac{d}{d\alpha}\left[v_1(\alpha)\sin\alpha\right] = \frac{1}{v_2^2(\beta)}\frac{d}{d\beta}\left[v_2(\beta)\sin\beta\right] = C.$$

We now desire to reach the same generalization of Snell's Law of Refraction using optimal control theory.

We first include the derivation of the general problem, using optimal control. For a further explanation of this process, see Pontryagin [Ref. 11]. It is required to join two points (x_1,y_1) and (x_2,y_2), see Figure 4.4, with a continuous,

piecewise smooth curve in such a way that the functional, $J[y] = \int_{x_1}^{x_2} F(x,y,y')dx$ reduces

to a minimum, where

$$F(x,y,y') = \begin{matrix} f_1(x,y,y'), \ if\ (x,y) \epsilon M_1 \\ f_2(x,y,y'), \ if\ (x,y) \epsilon M_2 \end{matrix}$$

$$define\ x^0 = \int_{x_1}^{x_2} F(\xi,y,y')d\xi, x^1 = x, x^2 = y, y' = u$$

From the maximum principle we have:

$$\frac{dx^0}{dt} = F(x,y,y'), \quad \frac{d\psi_o}{dt} = 0$$

$$\frac{dx^1}{dt} = 1, \qquad \frac{d\psi_1}{dt} = \frac{\partial F}{\partial x}$$

$$\frac{dx^2}{dt} = u = y', \qquad \frac{d\psi_2}{dt} = \frac{\partial F}{\partial y}$$

$H = \Psi_0 F(x,y,y') + \Psi_1 + \Psi_2 y'$

To maximize H, we set $\frac{\partial H}{\partial u} = 0$:

$$\frac{\partial H}{\partial u} = \psi_0 \frac{\partial F}{\partial u} + \psi_2 = 0, \; where \; u = y'$$

$$\psi_2 = -\psi_0 \frac{\partial F}{\partial u}$$

Setting H = 0 we have:

$$0 = \psi_0 F(x,y,u) + \psi_1 - (\psi_o \frac{\partial F}{\partial u})u$$

$$\psi_1 = -\psi_0 F(x,y,u) + \psi_0 (\frac{\partial F}{\partial y'})y'$$

From the jump conditions, $\psi^+ = \psi^- + \mu$ grad g (x,y):

$$\psi_0 \left[-f_2 + \frac{\partial f_2}{\partial y'}(y^+)' \right] = \psi_0 \left[-f_1 + \frac{\partial f_1}{\partial y'}(y^-)' \right] + \mu N'$$

or,

$$-f_2 + \frac{\partial f_2}{\partial y'}(y^+)' = -f_1 + \frac{\partial f_1}{\partial y'}(y^-)' + \mu N'$$

and,

$$-\psi_0 \frac{\partial f_2}{\partial y'} = -\psi_0 \frac{\partial f_1}{\partial y'} + \mu N^2$$

or,

$$\frac{\partial f_2}{\partial y'} = \frac{\partial f_1}{\partial y'} + \mu N^2$$

where (N^1, N^2) is a normal vector to the curve g(x,y) = 0 at the break point of the trajectory. Let Y' denote the slope of the tangent to the curve of g(x,y) at that point:

$$\frac{\frac{\partial f_2}{\partial y'} - \frac{\partial f_1}{\partial y'}}{f_2 - f_1 + \frac{\partial f_1}{\partial y'}(y^-)' - \frac{\partial f_2}{\partial y'}(y^+)'} = \frac{-1}{Y'}$$

from which:

$$\frac{\partial f_2}{\partial y'}Y' - \frac{\partial f_1}{\partial y'}Y' = -f_2 + f_1 \frac{-\partial f_1}{\partial y'}(y^-)' + \frac{\partial f_2}{\partial y'}(y^+)'$$

or,

$$f_2 + \frac{\partial f_2}{\partial y'}(Y' - (y^+)') = f_1 + \frac{\partial f_1}{\partial y'}(Y' - (y^-)') .$$

Since the velocity of the light particles is a function of direction, the corresponding integral is: $\int_{x_1}^{x_2} f(y') \sqrt{1+y'^2}\, dx$, where $f(y') = \frac{1}{v_1 \tan^{-1}(y')}$.

Applying the results derived above, we have the following conditions:

$$\frac{\partial \psi_1}{\partial t} = \frac{\partial F}{\partial x} = 0 \text{ and}$$

$$\frac{\partial \psi_2}{\partial t} = \frac{\partial F}{\partial y} = 0 .$$

Therefore, $\psi_1 = C_1$, and $\psi_2 = C_2$. Using this information, we now have:

$$\psi_2 = \frac{\partial F}{\partial y'} = C_2$$

and

$$\psi_1 = F - \frac{\partial F}{\partial y'} y' = C_1 .$$

By the continuity of ψ_1 we have:

$$f_2 - \frac{\partial f_2}{\partial y'}(y^+)' = f_1 - \frac{\partial f_1}{\partial y'}(y^-)' .$$

We rewrite the right hand side of this equation as:

$$f_1(y')\sqrt{1+y'^2} - \left[f_1'(y')\sqrt{1+y'^2} + \frac{y'}{\sqrt{1+y'^2}} f_1(y') \right] y'$$

or,

$$\frac{f_1(y')}{\sqrt{1+y'^2}} - y' f_1'(y')\sqrt{1+y'^2} .$$

Recalling that y′ = tan α and that:

$$f_1'(y') = \frac{\partial f_1}{\partial y'} = \frac{\frac{-dv_1}{dy'}}{v_1(\alpha)^2} = \frac{\frac{\frac{-dv_1}{d\alpha}}{\frac{dy'}{d\alpha}}}{v_1(\alpha)^2} = \frac{\frac{\frac{-dv_1}{d\alpha}}{\sec^2\alpha}}{v_1(\alpha)^2} = \frac{-\cos^2\alpha \frac{dv_1}{d\alpha}}{v_1(\alpha)^2} .$$

we then arrive at the expression:

$$\frac{\cos\alpha}{v_1(\alpha)} + \frac{\cos^2\alpha \frac{dv_1}{d\alpha}}{v_1(\alpha)^2} \tan\alpha \sec\alpha$$

which then becomes:

$$\frac{v_1(\alpha)\cos\alpha + \frac{dv_1}{d\alpha}\sin\alpha}{v_1(\alpha)^2}.$$

This is our familiar expression for this particular generalization of Snell's Law of Refraction:

$$\frac{1}{v_1(\alpha)^2}\frac{d}{d\alpha}\left[v_1(\alpha)\sin\alpha\right] = \frac{1}{v_2(\beta)^2}\frac{d}{d\beta}\left[v_2(\beta)\sin\beta\right].$$

5. Optically Anisotropic Media Without the use of Lagrange Multipliers

Consider again the same case where the velocity of the light particles depends upon the direction at each point. We wish to reach a generalization of Snell's Law without the use of Lagrange multipliers. A look at Figure 4.3 shows that we need to minimize the following expression:

$$T = \frac{a\sec\phi_1}{f(\phi_1)} + \frac{b\sec\phi_2}{g(\phi_2)}$$

where $r = f(\Phi_1)$ is the polar equation of a smooth starshaped curve and $r = g(\Phi_2)$ is similarly defined. A smooth starshaped curve has the property that every line segment connecting any point within the curve to the center of the curve remains completely within the curve. Another look at Figure 4.3 reveals that

$$C = a\tan(\Phi_1) + b\tan(\Phi_2).$$

Let $h(\Phi_1,\Phi_2) = a\tan(\Phi_1) + b\tan(\Phi_2) - C$. Recalling the definition of the total differential for a function, h, where $h = h(\Phi_1,\Phi_2)$, we have:

$$dh = \frac{\partial h}{\partial \phi_1} d\phi_1 + \frac{\partial h}{\partial \phi_2} d\phi_2$$

Then the following relationship holds:

$$\frac{d\phi_2}{d\phi_1} = \frac{\frac{-\partial h}{\partial \phi_1}}{\frac{\partial h}{\partial \phi_2}} = \frac{-a\sec^2(\phi_1)}{b\sec^2(\phi_2)}$$

Now since we wish to determine the minimum of a function we simply set the derivative of the function equal to zero, that is:

$$\frac{dT}{d\phi_1} = 0 .$$

This, and the chain rule give us:

$$\frac{\left[a\sec\phi_1\tan\phi_1 f(\phi_1) - a\sec\phi_1 f'(\phi_1)\right]}{f^2(\phi_1)} + \frac{\left[b\sec\phi_2\tan\phi_2 g(\phi_2)\frac{d\phi_2}{d\phi_1} - b\sec\phi_2 g'(\phi_2)\frac{d\phi_2}{d\phi_1}\right]}{g^2(\phi_2)} = 0$$

which further reduces to:

$$\frac{\left[a\sec\phi_1\tan\phi_1 f(\phi_1) - a\sec\phi_1 f'(\phi_1)\right]}{f^2(\phi_1)} = a\sec^2\phi_1 \frac{\left[\sin\phi_2 g(\phi_2) - \cos\phi_2 g'(\phi_2)\right]}{g^2(\phi_2)}$$

which yields:

$$\frac{\left[f(\phi_1)\sin\phi_1 - f'(\phi_1)\cos\phi_1\right]}{f^2(\phi_1)} = \frac{\left[g(\phi_2)\sin\phi_2 - g'(\phi_2)\cos\phi_2\right]}{g^2(\phi_2)} .$$

This expression is yet another representation of Snell's Law of Refraction for our given problem.

6. Optically Anisotropic Media in Two Dimensions

Let us now consider the problem of the propagation of light in a two dimensional anisotropic media. That is, consider a light particle traveling from a point A to a point B. Let the velocities of the light particles be some function of two spatial variables, say y and z : $v = v(y,z)$. For instance, suppose we arrange four glass containers, two on top of each other, and place different liquids in each container, resulting in varying particle velocities in each medium. See Figure 4.5.

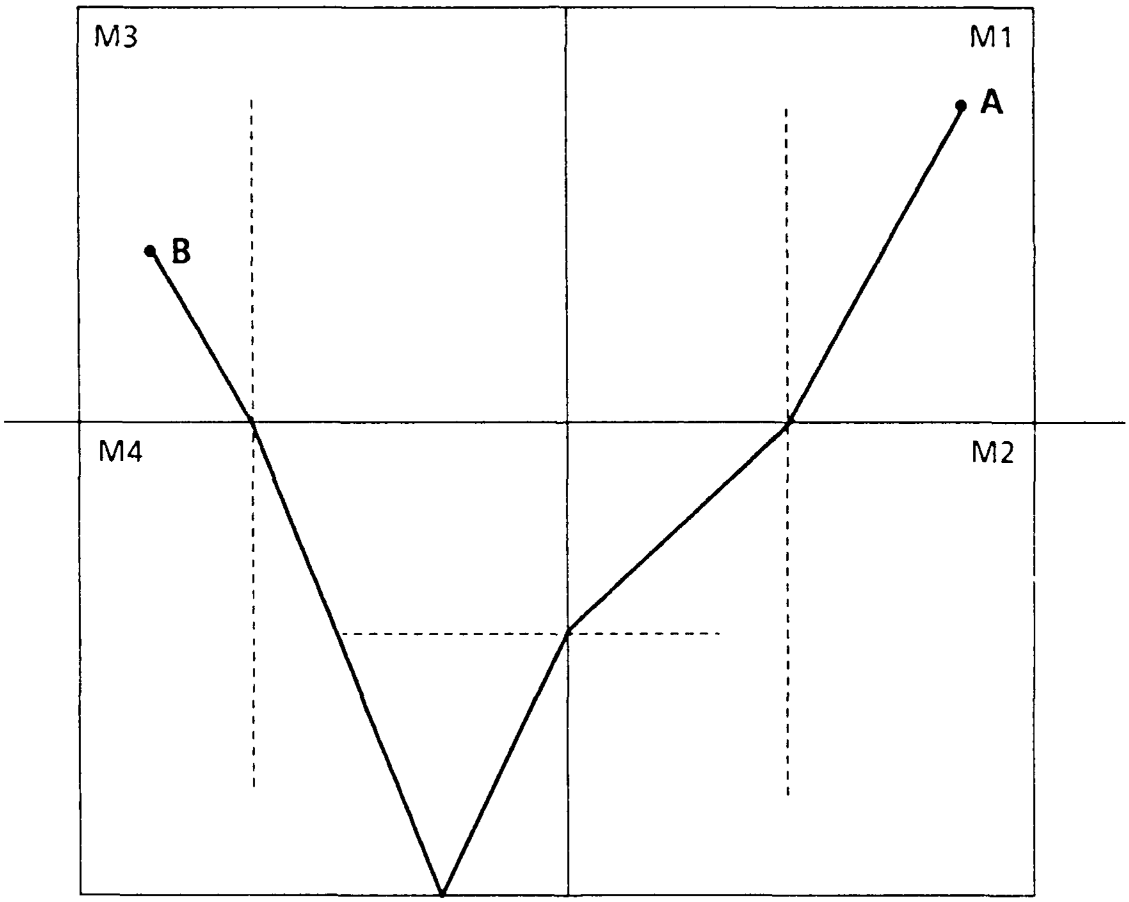

Figure 4.5 Propagation of Light in Two Dimensions

Applying Fermat's Principle, we arrive at our familiar expression using variational methods:

$$T = \int_A^B \frac{ds}{\frac{ds}{dt}} = \int_A^B \frac{\sqrt{1 + y'^2 + z'^2}}{v(y,z)} dx .$$

Our functional

$$J[y] = \int_A^B F(y, y', z, z',) dx$$

has no x dependence. Therefore, starting with the two Euler - Lagrange equations for our situation:

$$F_y - \frac{d}{dx}(F_{y'}) = 0$$

and

$$F_z - \frac{d}{dx}(F_{z'}) = 0 .$$

We again reduce these two second order differential equations as before and are left with:

$$F_y y' - F_{y'y} y'^2 - F_{y'y'} y' y'' = \frac{d}{dx}(F - y' F_{y'}) .$$

Thus, in this case, Euler's equation has the first integral:

$$F - y' F_{y'} = C$$

where C is a constant. Similarly for Z we arrive at:

$$F - z' F_{z'} = C.$$

Applying these two Euler-Lagrange equations to our functional we see that:

$$\frac{\sqrt{1+y'^2+z'^2}}{v(y,z)} - y'\frac{y'}{\sqrt{1+y'^2+z'^2}\ v(y,z)} = C_1$$

and,

$$\frac{\sqrt{1+y'^2+z'^2}}{v(y,z)} - z'\frac{z'}{\sqrt{1+y'^2+z'^2}\ v(y,z)} = C_2$$

which yield:

$$\frac{1+z'^2}{\sqrt{1+y'^2+z'^2}\ v(y,z)} = C_1$$

and,

$$\frac{1+y'^2}{\sqrt{1+y'^2+z'^2}\ v(y,z)} = C_2 .$$

Dividing the above two expressions and letting $\frac{C_1}{C_2} = C_3$ we arrive at:

$$\frac{1+z'^2}{1+y'^2} = C_3$$

or,

$$1 + z'^2 = C_3 (1 + y'^2).$$

If we let $z' = \tan(\alpha)$ and let $y' = \tan(\beta)$ where $\phi_1 = \frac{\pi}{2} - \alpha$ *and* $\phi_2 = \frac{\pi}{2} - \beta$:

$$1 + \tan^2(\alpha) = C_3 (1 + \tan^2 (\beta))$$

or,

$$\sec^2(\alpha) = C_3 \sec^2 (\beta)$$

or letting $C_4 = \sqrt{C_3}$

$$\sec(\alpha) = C_4 \sec(\beta)$$

which in turn equates to:

$$\cos\alpha = C_5 \cos\beta \textit{ where } C_5 = \frac{1}{C_4}.$$

Remembering that $\phi_1 = \frac{\pi}{2} - \alpha \textit{ and } \phi_2 = \frac{\pi}{2} - \beta$ we now have the familiar form of

Snell's Law:

$$\sin(\Phi_1) = C_5 \sin(\Phi_2),$$

where C_5 represents the ratio of the velocities in the different media.

V. APPLICATIONS

A. INTRODUCTION

In all of our previous derivations of Snell's Law of Refraction, our functionals are of the form:

$$J[y] = \int_a^b f(x,y)\sqrt{1+y'^2}\, dx .$$

This represents the integral of a function, f(x,y) with respect to the arc length s, where $ds = \sqrt{1+y'^2}\, dx$. In this case, the Euler-Lagrange equation can be reduced to the following:

$$f_y - f_x y' - \frac{f y''}{1+y'^2} = 0$$

where f = f(x,y). An explanation of this derivation is found in Gelfand and Fomin [Ref. 6:p. 19]. In our various developments of Snell's Law, the function f(x,y), represents the reciprocal of the velocity of the light particles in question. Recalling from physics that:

$$\frac{\sin\phi_1}{\sin\phi_2} = \frac{v_1}{v_2} = n_{12} \; [Ref. 1]$$

we now introduce the term, index of refraction: n_{12}. Thus, in all of our earlier discussions we employ a functional which involves the index of refraction. In fact we have n(x,y) = f(x,y). With this in mind, we now develop certain inverse

problems to illustrate the importance of the index of refraction to various acoustical and optical problems.

B. INVERSE PROBLEMS

We pose the following problem:

Given a certain host medium, find the index of refraction which will allow a particle of light to travel from a point A at $(0,0)$ to a point B at $(1,1)$ or to a point C at $(\frac{\pi}{2} - 1, 1)$ along the following curves: (1) a parabola (concave down), (2) a parabola (concave up), (3) a circle, (4) a cycloid, and (5) a straight line. See Figure 5.1.

1. Parabola

We solve these different problems in order, starting with the parabolas. Since the parabola must have the following endpoints $y(0) = 0$, and $y(1) = 1$ we necessarily need the parabola: $y = x^2$. Applying Euler's equation discussed above we arrive at:

$$-n_x y' = \frac{n y''}{1 + y'^2}$$

or,

$$-n_x(2x) = \frac{2n}{1 + 4x^2}$$

$$\frac{-n_x}{n} = \frac{1}{x(1 + 4x^2)}$$

which, by partial fraction decomposition becomes:

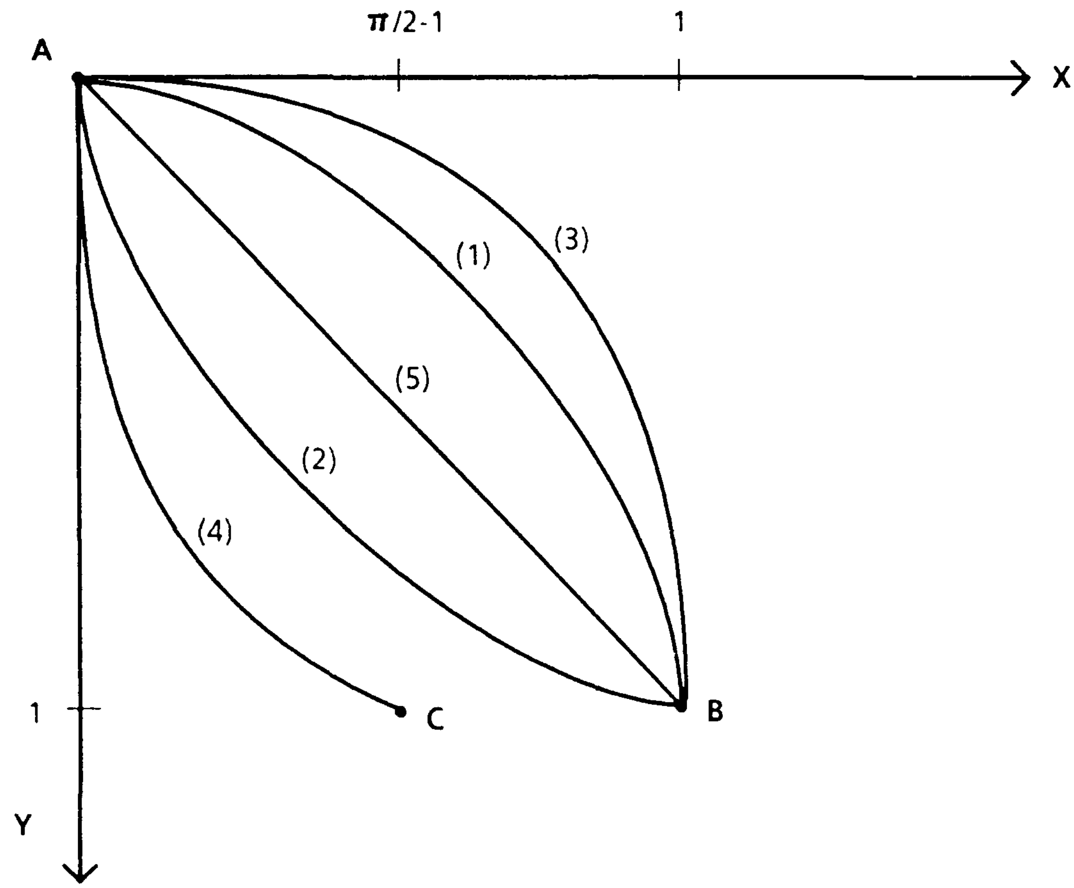

Figure 5.1 Particle Paths in an Isotropic Medium

$$\frac{-n_x}{n} = \frac{1}{x} - \frac{4x}{1+4x^2}.$$

This integrates to yield:

$$\ln|n| = -\ln|x| + \frac{1}{2}\ln|1+4x^2| + \ln|C|.$$

Therefore, the desired index of refraction, is:

$$n(x) = \frac{C\sqrt{1+4x^2}}{x}.$$

The other parabola (concave up) also requires the same endpoint restrictions, y(0) = 0 and y(1) = 1. Therefore, we force the equation of our parabola to be $y = \sqrt{x}$. Applying Euler's equation, we are left with:

$$-n_x y' = \frac{ny''}{1+y'^2}$$

$$-n_x \frac{1}{2\sqrt{x}} = \frac{n\frac{-1}{4\sqrt{x^3}}}{1+\frac{1}{4x}}$$

or,

$$-n_x \frac{1}{2\sqrt{x}} = \frac{-n}{\sqrt{x}\,(4x+1)}$$

from which it follows that:

$$\frac{n_x}{n} = \frac{2}{4x+1}$$

which, after integration becomes:

$$\ln|n| = \frac{1}{2}\ln|4x+1| + \ln|C|.$$

So, the desired index of refraction for this parabola is the following:

$$n(x) = C\sqrt{4x+1}\,.$$

2. Circle

Now, we consider the circular path. Again, we desire that our curve pass through the points A and B. Therefore, our equation for the required circle is:

$$x^2 + (y-1)^2 = 1$$

which yields:

$$y = \sqrt{1-x^2} + 1\,.$$

From our earlier discussion, the Euler equation is:

$$n_y - n_x y' - \frac{n y''}{1+y'^2} = 0$$

from which we get:

$$n_x \frac{x}{\sqrt{1-x^2}} = \frac{n\left[\frac{-x^2}{\sqrt{(1-x^2)^3}} - \frac{1}{\sqrt{1-x^2}}\right]}{1+\frac{x^2}{1-x^2}}$$

or,

$$n_x \frac{x}{\sqrt{1-x^2}} = \frac{-n(1-x^2)}{\sqrt{(1-x^2)^3}}$$

from which follows:

$$n_x \frac{x}{\sqrt{1-x^2}} = \frac{-n}{\sqrt{1-x^2}}$$

or,

$$\frac{n_x}{n} = \frac{-1}{x}.$$

Therefore, after integrating we arrive at:

$$\ln|n| = -\ln|x| + \ln|C| = \ln|\frac{1}{x}| + \ln|C|.$$

Hence, the desired index of refraction for this medium, in order to travel along a circle from point A to point B is:

$$n(x) = \frac{c}{x}.$$

3. Cycloid

Recalling from differential equations, the parametric equations of a cycloid are of the form:

$$x = \frac{k^2}{2}(\phi - \sin\phi)$$

$$y = \frac{k^2}{2}(1 - \cos\phi) \quad [Ref.\ 13].$$

Since we desire that our cycloid pass through the points A and C we necessarily reduce the above equations to:

$$x = \Phi - \sin\Phi$$

$$y = 1 - \cos\Phi.$$

By trial and error we determine that the function associated with a cycloid is only a function of y. As such, the corresponding Euler equation is:

$$n_y = \frac{ny''}{1+y'^2}$$

or,

$$\frac{n_y}{n} = \frac{1}{2}\,\frac{2y''}{1+y'^2}\,.$$

The chain rule and integration yield:

$$\int_A^C \frac{n_y}{n}dy = \int_A^C \frac{y''}{1+y'^2}dy = \int_A^C \frac{y''}{1+y'^2}\,\frac{dy}{dx}\,dx$$

or,

$$\ln|n| = \frac{1}{2}\ln|1+Y'^2| + \ln|C|$$

so,

$$n(y) = C\sqrt{1+y'^2}\,.$$

Now since $\frac{dy}{dx} = \frac{\frac{dy}{d\phi}}{\frac{dx}{d\phi}} = \frac{\sin\phi}{1-\cos\phi}$ we have:

$$n(y) = C\sqrt{\frac{1+\sin^2\phi}{(1-\cos\phi)^2}}$$

or,

$$n(y) = C\sqrt{\frac{2(1-\cos\phi)}{(1-\cos\phi)^2}}\,.$$

Therefore, the desired index of refraction for the cycloid is:

$$n(y) = C\sqrt{\frac{2}{y}}\,.$$

4. Straight Line

Our final problem involves the path of a straight line from point A to point B. That is, y = x. Using the Euler equation, we arrive at:

$$n_y - n_x y' - \frac{ny''}{1+y'^2} = 0$$

$$-n_x y' = \frac{ny''}{1+y'^2}$$

$$n_x = 0$$

which integrates to yield the following:

$$n(x) = C.$$

Therefore, the index of refraction in the case of a straight line path is:

$$n(x) = C.$$

Computing the various indices of refraction does indeed give us the desired curve from a point A to a point B. For sake of brevity, we consider the first case, that of the parabola (concave down). That is, we show that:

$$\min J[y] = \int_0^1 \frac{\sqrt{1+4x^2}}{x} \sqrt{1+y'^2}\, dx$$

will yield the curve $y(x) = x^2$.

Applying the usual Euler-Lagrange equation, that is:

$$F_y - \frac{d}{dx} F_{y'} = 0, \textit{ yields:}$$

$$-\frac{d}{dx}\left[\frac{\sqrt{1+4x^2}}{x} \frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

or,

$$\frac{\sqrt{1+4x^2}}{x}\frac{y'}{\sqrt{1+y'^2}} = C$$

from which it follows that:

$$\frac{y'^2}{1+y'^2} = \frac{c^2x^2}{1+4x^2}$$

or,

$$y'^2 = \frac{c^2x^2}{1+(4-c^2)x^2}$$

Setting C = 2 yields:

$$y' = 2x$$

or,

$y(x) = x^2$ which is the equation for the parabola (concave down) passing from point A to point B in our problem.

C. GENERAL EXPRESSIONS FOR THE INVERSE PROBLEM

1. Function of X Only

It is possible to derive a single, generic expression for a family of curves in terms of any function, f(x,y), where the various functions f(x,y) determine the type of curve generated. We now show the development of such an expression. We begin with our standard form of our functional:

$$J[y] = \int_a^b f(x,y)\sqrt{1+y'^2}\,dx .$$

We first restrict our attention to those functions, f of the form $f = f(x)$, that is, those functions that are only dependent upon x. This leaves us with:

$$J[y] = \int_a^b f(x)\sqrt{1+y'^2}\,dx .$$

Now, since the integrand does not depend on y, the corresponding Euler equation is:

$$F_{y'} = C$$

or,

$$\frac{f(x)y'}{\sqrt{1+y'^2}} = C$$

which further reduces to:

$$y' = \frac{C}{\sqrt{f(x)^2 - C^2}} .$$

Now integrating the expression yields:

$$y + \alpha = \int \frac{c\,dx}{\sqrt{f(x)^2 - c^2}} .$$

This, then is the simple expression we have been looking for when the function f(x,y) is only dependent upon x. With this expression, we can now input any function and in turn generate a family of curves. As an example, we make use of an earlier problem, that of the path of a circle. In that development, we saw that the corresponding function for a circular path was:

$$f(x) = \frac{C}{x}.$$

For simplicity, we will let C = 1. That is, our problem involves $f(x) = \frac{1}{x}$ and we

wish to show that this will generate a family of circles.

Starting with the simple expression we have:

$$y = \int \frac{c\,dx}{\sqrt{f(x)^2 - C^2}}$$

which becomes:

$$y = \int \frac{c\,dx}{\sqrt{\frac{1}{x^2} - c^2}}$$

or,

$$y = \int \frac{cx\,dx}{\sqrt{1 - c^2x^2}}.$$

Letting u = $1 - C^2x^2$ we have:

$$y = \frac{-1}{2}C\int \frac{du}{\sqrt{u}}$$

which becomes:

$$y - C_1 = \frac{-1}{C\sqrt{1 - C^2x^2}}$$

or,

$$(y - C_1)^2 + x^2 = \frac{1}{C^2}$$

which is the equation of a circle, centered at $(0,C_1)$ with radius $\sqrt{\frac{1}{C^2}}$. This procedure can be followed for any function and will generate a family of curves associated with that particular function.

2. Function of Y Only

We now turn our attention to those functions which are dependent upon y alone. We derive an expression which will also produce a family of curves. As before, we start with the now familiar functional:

$$J[y] = \int_a^b f(x,y)\sqrt{1 + y'^2}\, dx .$$

We consider only those functions dependent on y:

$$J[y] = \int_a^b f(y)\sqrt{1 + y'^2}\, dx .$$

Since the integrand has no explicit dependence on x, we employ the following Euler equation:

$$F - y'F_{y'} = C.$$

So, we have:

$$f(y)\sqrt{1 + y'^2} - y'\frac{f(y)y'}{\sqrt{1 + y'^2}} = C$$

or,

$$f(y)[1+y'^2] - \frac{f(y)y'^2}{\sqrt{1+y'^2}} = C$$

which reduces to:

$$f(y) = C\sqrt{1+y'^2}$$

or,

$$y' = \sqrt{\frac{f(y)^2 - C^2}{C^2}}.$$

After integration this becomes:

$$x + \alpha = \int \frac{c\,dy}{\sqrt{f(y)^2 - c^2}}.$$

This then is the expression which produces any family of curves dependent only upon the function f(y). We now show that this expression, when given the proper function f(y), yields the family of curves for a cycloid. Recalling the earlier discussion, the corresponding function is $f(y) = \frac{C\sqrt{2}}{\sqrt{y}}$. For simplicity we let $C = \frac{1}{\sqrt{2}}$.

This reduces the function to $f(y) = \frac{1}{\sqrt{y}}$. Plugging this into the earlier derived expression, we have:

$$x + \alpha = \int \frac{c\,dy}{\sqrt{\frac{1}{y} - c^2}}.$$

This leaves us with a Volterra integral equation of the first kind:

$$x + \alpha = C\int_0^y \frac{d\eta}{\sqrt{\frac{1}{\eta} - c^2}}.$$

For a further explanation and development of the Volterra integral equation, see Keener [Ref. 25:p. 101]. Using the Fundamental Theorem of Calculus and differentiating both sides with respect to y, we are left with:

$$\frac{dx}{dy} = \frac{C}{\sqrt{\frac{1}{y - C^2}}}$$

or,

$$\frac{dy}{dx} = \frac{1}{C}\sqrt{\frac{1}{y - C^2}}.$$

This reduces to:

$$y'^2 + 1 = \frac{1}{C^2 y}$$

or,

$$y(y'^2 + 1) = \frac{1}{C^2}.$$

This differential equation represents the equation of a family of cycloids [Ref. 27]. We include a solution to this differential equation for completeness. Let $\frac{1}{C^2} = k^2$ and let $y = k^2\sin^2 t$. Then, we are left with the following expression:

$$(1 + y'^2)k^2\sin^2 t = k^2$$

or,

$$1+y'^2=\frac{1}{\sin^2 t}.$$

This reduces to:

$y' = \cot(t)$. We consider only the positive square root in this case, since the slope of the graph is positive here. This then yields:

$$dx=\frac{dy}{\cot(t)}=\tan(t)\,dy.$$

However, $dy = k^2(2\sin(t)\cos(t))dt$. Thus,

$$dx = 2k^2\sin^2 t\,dt.$$

We now let $\Phi = 2t$, apply the initial conditions, and make use of the double angle formula, $\sin^2 t=\frac{1-\cos\phi}{2}$, to arrive at:

$$dx=\frac{k^2}{2}(1-\cos\phi)\,d\phi$$

$$dy=\frac{k^2}{2}(\sin\phi)\,d\phi.$$

This yields:

$$x=\frac{k^2}{2}(\phi-\sin\phi)+C_1$$

and,

$$y=\frac{k^2}{2}(1-\cos\phi)+C_2.$$

Using the fact that at $\Phi = 0$, $x = 0$ and $y = 0$ yields:

$$x = \frac{k^2}{2}(\phi - \sin\phi)$$

and,

$$y = \frac{k^2}{2}(1 - \cos\phi)\,.$$

These are the equations of a family of cycloids. Again, this procedure can be followed for any function f(y) and will yield corresponding families of curves.

VI. CONCLUSIONS

This thesis presents several derivations of Snell's Law of Refraction and one of the principle of reflection. In all of these, we place different constraints upon the problem and produce the familiar laws governing reflection and refraction. The main result of this research is the expression found for Snell's Law of Refraction in which the velocity of the light particles in the medium depends upon direction. We also use inverse problems to find the index of refraction needed to move along any desired family of curves. Finally, we derive a general expression that generates a family of curves given any index of refraction.

This research has several applications in the area of acoustical and optical wave propagation. In the area of acoustics, the inverse problems enable us to predict the path followed by sound waves for a given medium. This would allow for military use in the area of prosecuting submarine contacts and in ocean bottom surveys. The inverse problems also have applications to the bending of light rays used in laser technology.

Although this paper derives several generalizations of Snell's Law of Refraction there are still areas that require further research. For instance, consider the inverse problem in space where the index of refraction is a function of three variables rather than two. Also, in all of the derivations in this paper the Hamilton-Jacobi equations

were not used. These equations, from the calculus of variations, could be used to derive a generalization of Snell's Law of Refraction.

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